Modeling Martin Löf Type Theory in Categories

François Lamarche

Modelling Martin Löf Type Theory in Categories

François Lamarche

INRIA Saclay – Ile de France, Equipe-Projet Parsifal

Abstract

We present a model of Martin-Löf type theory that includes both dependent products and the identity type. It is based on the category of small categories, with cloven Grothendieck bifibrations used to model dependent types. The identity type is modelled by a path functor that seems to have independent interest from the point of view of homotopy theory. We briefly describe this model's strengths and limitations.

1. Introduction

The last few years have seen a flurry of activity in the semantics of Martin-Löf's identity type, based on the fruitful relationship with path objects in homotopy theory.

In this paper we present such a model of identity types in Martin-Löf type theory which has both desirable features of including dependent products and having introduction-elimination operators that are stable under substitution. Moreover our presentation is very concrete, and calculations in the model are fairly easy; in particular no use whatsoever is made of factorization systems, which have been a favored technique in the semantics of identity types [1, 19].

This work was first presented at the Makkaifest in Montreal in June 2009. Among the other models that were being developed contemporaneously or semi-contemporaneously, one deserves special mention [5]. Not only does the construction of the simplicial path object described in that paper very much resemble ours—this is not a big surprise, since our model is built on small categories, and a category is a special kind of simplicial set—but also one of its important ingredients is what we have called a triangulator in the present paper. One additional interesting aspect of [5] is that an axiomatic framework is presented for path objects. Our own model almost fits that framework, but not quite, which suggests the existence of a more general framework, that would encompass both approaches. This axiomatic framework also suggests a way to obtain the present model (or something very close to it) by the means of a factorization system.

Email address: `francois.lamarche@inria.fr` (François Lamarche)

Preprint submitted to Elsevier *September 19, 2013*

Independently of its type-theoretical interest, our model seems to have significant interest in the homotopy theory of categories, a subject we intend to investigate.

2. What we are looking for

What follows is the structure we will require on a category $\mathscr{C}$ in order to get a model of dependent type theory with an identity predicate. There is nothing original here, except that the presentation is optimized for our purposes.

The first categorical models of dependent types [2] relied on a class of maps $\mathscr{F}$ of $\mathscr{C}$ with the following properties.

- $\mathscr{C}$ has a terminal object,
- All isos in $\mathscr{C}$ are in $\mathscr{F}$,
- $\mathscr{F}$ is closed under pullbacks by arbitrary morphisms of $\mathscr{C}$.

We will try to use the following notation consistently. We try as much as we can to denote a map X of $\mathscr{F}$ as something like

$$X : \overline{X} \longrightarrow A \,,$$

where the name of its domain is obtained by overlining the map's name, but the codomain can look arbitrary. Given the above along with an arbitrary $f : B \to A$ the pullback operation is denoted by

$$\begin{tikzcd} \overline{f^*X} \arrow[r] \arrow[d, "f^*X"'] & \overline{X} \arrow[d, "X"] \\ B \arrow[r, "f"'] & A \end{tikzcd} \tag{1}$$

The members of $\mathscr{F}$ are most often called *display maps* but we prefer to call them *abstract fibrations,* and we will often just say *fibrations* when the context is clear.

The intuition should be clear and has been used in geometry since the early fifties: given X as above, it is thought of as dependent family $(X_a)_{a\in A}$, and in a concrete category of sets-with-structure the X_a are just the fibers $X^{-1}(a)$. The pullback operation corresponds to substitution: given f as above, then f^*X models the family $(X_{f(b)})_{b\in B}$.

The maps of $\mathscr{F}$ that have the terminal object $\mathbf{1}$ as codomains correspond to ordinary, non-dependent types. In many models all maps to the terminal are in $\mathscr{F}$, but this does not have to be the case in general. Notice that since syntactic entities are built by an inductive process that starts with non-dependent types, the only objects A of $\mathscr{C}$ that appear in the interpretation of a syntactical system are those for which there is a chain $A \to \cdot \to \cdots \to \mathbf{1}$ of display maps to the terminal object. Abstract fibrations whose codomain is a terminal will be just denoted by the source of the domain, since they just correspond to objects of $\mathscr{C}$ and the overline notation becomes cumbersome.

Let $\mathbf{2}$ denotes the category with two objects and one arrow between them. Thus $\mathscr{C}^{\mathbf{2}}$ is the familiar category whose objects are maps and whose arrows are commutative squares. It is profitable to think of $\mathscr{F}$ as the full subcategory of $\mathscr{C}^{\mathbf{2}}$, whose objects are the abstract fibrations. The axiom of stability under pullbacks means that the composite

$$\mathscr{F} \longrightarrow \mathscr{C}^{\mathbf{2}} \xrightarrow{\text{Cod}} \mathscr{C} \tag{2}$$

of the inclusion functor with the codomain functor is a "large" Grothendieck fibration[1]. In this context it is natural to call the maps of abstract fibrations that are pullback squares *Cartesian* maps or squares. In the ordinary world of independent type, the categorical version of a unary type constructor is just an endofunction on the class of objects of the modeling category. In the world of dependent types, a unary type constructor is an endofunctor on the category of fibrations and cartesian maps. A type constructor which is a covariant functor is an endofunctor on the category of fibrations and all squares which also sends cartesian maps to cartesian maps. A contravariant type constructor is something a little more elaborate.

To get a completely formalized interpretation of type theory it should be required that the pullback operation be functorial, instead of pseudo-functorial as is the case for ordinary pullbacks in a category. But this requirement is completely independent of the rest and can always be obtained by massaging the target category properly [6], and no further mention of this condition will be made in this paper.

The pair $(\mathscr{C}, \mathscr{F})$ is said to have *dependent products* when the following further condition is obeyed.

- For any $X \in \mathscr{F}$ the pullback functor X^* has a right adjoint, which we denote Π_X, with the Beck-Chevalley condition holding.

Let us recall the Beck-Chevalley condition: take an arbitrary pullback square of fibrations, as in Equation (1), denoting the upper horizontal arrow by h, and let $Y: \overline{Y} \to \overline{X}$ be another fibration. Beck-Chevalley means that the natural morphism

$$f^*(\Pi_X Y) \longrightarrow \Pi_{f^*X}(h^*Y)$$

obtained by

$$\frac{\dfrac{h^*X^*(\Pi_X Y) \xrightarrow{h^*\epsilon} h^*Y}{(f^*X)^* f^*(\Pi_X Y) \longrightarrow h^*Y}}{f^*(\Pi_X Y) \longrightarrow \Pi_{f^*X}(h^*Y)}$$

[1]For the unitiated reader, the formal definition of Grothendieck fibrations, cartesian maps and cleavages is given at the beginning of Section 3

has to be an isomorphism. This allows us to model the operator Π from Martin-Löf type theory.

The Beck-Chevalley condition is something which is completely invisible in syntax, but has to be made explicit in a categorical context if we want a real correspondence between the two.

The structure above is very useful, but not all models of dependent type theory fit into it. We need to add a little more. In general we will model types by *pairs* (X, Φ), where X is an abstract fibration just as above, and Φ a little extra something which we can call a *structure.* Since for us the base category $\mathscr{C}$ will always be Cat, and the abstract fibrations will be some kind of Grothendieck fibrations between small categories (actually Grothendieck *bi*fibrations) we can think of a structure Φ in this context as a *cleavage* for the fibration X, an algebraic structure which gives "constructive witness" to the property of being a fibration.

By defining a map between two pairs $(Y, \Psi) \to (X, \Phi)$ as just a commutative square, (i.e., ignoring the structures) we get a category $\mathscr{S}$ and a forgetful functor $\mathscr{S} \to \mathscr{F}$ which is obviously an equivalence. We require that $\mathscr{S}$ contain a specific subcategory whose maps we call *cartesian,* with the following important property:

- Given a pullback square as below and a structure Φ for X

$$\begin{tikzcd} \overline{Y} \arrow[r, "g"] \arrow[d, "Y"'] & \overline{X} \arrow[d, "X"] \\ B \arrow[r, "f"'] & A \end{tikzcd} \tag{3}$$

  then there is a unique structure $f^*\Phi$ on Y that makes the pair (f, g) a cartesian map $(Y, f^*\Phi) \to (X, \Phi)$.

An immediate consequence of this property is that the composite $\mathscr{S} \to \mathscr{F} \to \mathscr{C}$ is a "large" Grothendieck fibration too, and that $\mathscr{S} \to \mathscr{F}$ is a cartesian functor of fibrations, which is a discrete fibration when restricted to cartesian maps, because of the uniqueness condition on pullbacks of structures. Any reader who is familiar with the standard semantical technology of dependent types should be able to convert the above prescription into a *full comprehension category* [8] or a *type category* [14]. Our main departure from these standard approaches has to do with viewpoint and notation: we consider types primarily as *maps,* but equipped with *extra structure*, which we have written so far as something like (X, Φ). But since it will happen very seldom that we have to consider more than two structures on the same map X, we will tend to write the structure as a decorated version $\underline{X}$ of the map X. This is a common form of abuse of notation.

The more standard approach in the literature is to consider the objects of $\mathscr{S}$ as the primary objects of interest, and name their associated underlying maps by applying to them additional notation for the projection functor $\mathscr{S} \to \mathscr{C}^{\mathbf{2}}$. Given $(X, \Phi) \colon \overline{X} \to A$ we will say that (X, Φ) is a *type in context (or base) A.*

Thus, we can say that we are dealing with three kinds of maps, that more or less live in $\mathscr{C}$. There are ordinary maps, fibrations, and fibrations-with-a-chosen-structure, i.e., types. All three can coexist in the same diagram. The notion of commutation for a diagram involving types just coincides with that of their underlying fibrations. The only additional concept which has to be added for dealing with types is that of a "pullback square" for which two parallel maps are types.

To reduce clutter, in some diagrams we will notate a type of the form $(X, \underline{X}) \colon \overline{X} \to A$ with just a thick arrow: $X : \overline{X} \longrightarrow A$.

Now, the definition of dependent products has to be adapted to this more general context. It just amounts to requiring that given

$$\overline{Y} \xrightarrow{(Y,\Psi)} \overline{X} \xrightarrow{(X,\Phi)} A$$

the adjoint to pulling back fibrations $\Pi_X Y$ defined above exists, and in addition that there is a structure $\Pi_\Phi \Psi$ obtained from Φ, Ψ that makes $(\Pi_X Y, \Pi_\Phi \Psi)$ a type.

2.1. Path objects and identity types

Martin-Löf's syntactic rules for identity type follow the standard pattern of natural deduction: one type constructor with the term corresponding to its introduction rule, and one term constructor for the elimination rule of the type. These requirements, formulated in the language presented above, are as follows. We want a type constructor that will map every type $(X, \underline{X})$ of $\mathscr{S}$, with $X \colon \overline{X} \to A$ to a type $(\mathbf{P}X, \underline{\mathbf{P}X})$ where

$$\overline{\mathbf{P}X} \xrightarrow{\mathbf{P}X} X \times_A X$$

is also denoted sometimes

$$\overline{\mathbf{P}X} \xrightarrow{\langle \partial_0 X, \partial_1 X \rangle} X \times_A X$$

which is equipped with a map $\mathsf{r}X$ for the introduction rule

$$\begin{tikzcd} & \overline{\mathbf{P}X} \arrow[d, "\mathbf{P}X"] \\ X \arrow[ur, "\mathsf{r}X"] \arrow[r, "\Delta"'] & X \times_A X \end{tikzcd}$$

making the triangle above commute. This constructor is required to be stable under pullbacks in the following sense: let $f \colon B \to A$ in $\mathscr{C}$ be any map, and $g \colon \overline{f^*X} \to \overline{X}$ the top map of the pullback square, calling $g \times_X g \colon f^*X \times_B f^*X \longrightarrow X \times_A X$ (for lack of a better name) the obvious "doubled" version of that map.

Then the triangle associated to the type $\mathbf{P}f^*X$ (i.e., $(\mathbf{P}f^*X, \underline{\mathbf{P}f^*X})$, along with Δ_{f^*X} and r^*X) is the pullback of the triangle pictured above by $g \times_X g$ (it is easy to see that the diagonal is always stable by such pullbacks.)

It will turn out that in our model, given a fibration with structure (X, Φ) the pair $(\mathbf{P}X, \underline{\mathbf{P}X})$ does not actually depend on Φ, only on X. This simplifies notation considerably and makes our abuses of notation not that abusive.

The type constructor $\mathbf{P}$ is actually covariant functorial, but the additional structure can be obtained from the rest of the axioms we are presenting here.

The other ingredient is a term constructor J for elimination. Given a type $(X, \underline{X})\colon \overline{X} \to A$, the map $J_X(Z,t)$ is defined for every type $(Z, \underline{Z})$ where $Z\colon \overline{Z} \to \overline{\mathbf{P}X}$ and $t\colon X \to Z$ makes the left triangle below commute.

$$\begin{tikzcd}
 & \overline{Z} \arrow[d, "Z"'] \\
\overline{X} \arrow[ur, "t"] \arrow[r, "\mathsf{r}X"'] & \overline{\mathbf{P}X} \arrow[u, bend right, "J_X(Z,t)"']
\end{tikzcd} \tag{4}$$

Moreover $J_X(Z,t)$ is an extension of t, i.e., $J_X(Z,t) \circ \mathsf{r}X = t$, and it is a section of Z, i.e., $Z \circ J_X(Z,t)$ is the identity. This constructor is also required to be stable under change of base: let $f\colon B \to A$ be any map and from it constuct g and $g \times_X g$ as just above. The latter defines another map $h\colon \overline{\mathbf{P}f^*X} \to \overline{\mathbf{P}X}$ by pullback. Then given Z, t also as just above,

$$\begin{tikzcd}
\overline{h^*Z} \arrow[rrr] \arrow[ddd, "h^*Z"] & & & \overline{Z} \arrow[ddd, "Z"'] \\
 & \overline{f^*X} \arrow[ul, "t'"'] \arrow[r, "g"] \arrow[ddl, "\mathsf{r}f^*X"] & \overline{X} \arrow[ur, "t"'] \arrow[ddr, "\mathsf{r}X"'] & \\
 & & & \\
\overline{\mathbf{P}f^*X} \arrow[rrr, "h"'] \arrow[uuu, bend left, "J'"] & & & \overline{\mathbf{P}X} \arrow[uuu, bend right, "J_X(Z,t)"']
\end{tikzcd}$$

we can construct the pullback of Z by h and define the maps t' and J' as the result of "pulling back" t and $J_X(Z,t)$ along that square. The stability requirement is that

$$J_{f^*X}(h^*Z, t') = J' \ .$$

3. Our Fibrations

Let us begin by recalling some well-known stuff. First, let $X\colon \overline{X} \to A$ be a map of categories. Given objects $x \in \overline{X}$ and $a \in A$ we say, following custom, that x *is above* a to mean $X(x) = a$, and the same goes with maps.

Definition 1. *A map $s\colon y \to x$ of $\overline{X}$, which is above $m\colon b \to a$ is said to be* hypercartesian *(which in this paper we will shorten to* cartesian*) if it has the property that*

- *for every $n\colon c \to b$ in A and every $r\colon z \to x$ which is above mn, there is a unique $t\colon z \to y$ above n that makes the triangle in $\overline{X}$ commute.*

The map of categories X is a fibration *if for any pair (m, x) where $m\colon b \to a$ is in A and x above a there exists a cartesian map above m. A* cleavage *is a choice of a cartesian map for all such (m, x). A map $s\colon x \to y$ of $\overline{X}$ is* cohypercartesian *(but we will shorten to* cocartesian *here) if it is cartesian in the dual $X^{op}\colon \overline{X}^{op} \to A^{op}$. If X^{op} is a fibration, then we say X is an* opfibration*; the dual concept of a cleavage is called a* cocleavage*.*

One of the most standard examples of fibrations is as follows. Take a category $\mathscr{C}$ that has pullbacks. Then the codomain projection $\mathscr{C}^{\mathbf{2}} \to \mathscr{C}$, as seen in Equation (2) *is a fibration (here we are vague about set-theoretical size considerations). The domain projection is always an opfibration, whatever $\mathscr{C}$ is.*

The map of categories X is said to be a bifibration *if it is both a fibration and an opfibration. By the axiom of choice any bifibration can be equipped with a* bicleavage*, that is, the choice of both a cleavage and a cocleavage.*

Now back to the model we are constructing. As we have said before, our base category $\mathscr{C}$ will be the category Cat of small categories. The abstract fibrations will be Grothendieck bifibrations, and a choice of structure Φ for an abstract fibration (concretely, a bifibration) will be a bicleavage. Let us go through this again, so as to define the notation. To have an abstract fibration structure Φ on X is to have both a fibration half:

- for every pair m, x where $m\colon b \to a$ is a map in A and x an object of $\overline{X}$ above a, a map
$$m^*x \xrightarrow{\Phi^*_m(x)} x$$
above m that has the property that

- for every $n\colon c \to b$ in A and every $r\colon z \to x$ which is above mn, there is a unique
$$z \xrightarrow{\Phi^*_m(n;r)} m^*x$$
above n that makes the triangle in $\overline{X}$ commute.

and an opfibration half

- for every pair m, x where $m\colon a \to b$ is a map in A and x an object of $\overline{X}$ above a, a map
$$x \xrightarrow{\Phi^m_*(x)} m_*x$$
such that

- for every $n\colon b \to c$ in A and every $r\colon x \to z$ which is above $m \circ n$, there is a unique
$$m_*x \xrightarrow{\Phi^m_*(n;r)} z$$

above n making the triangle in $\overline{X}$ commute[2].

The reader should take note that if m above is an identity, the maps $\Phi^*_m(x)$ and $\Phi^m_*(x)$ are not required to be identity maps, although they are necessarily isos. This is actually a desirable feature for the semantics. Obviously, the (bi)fibration structure Φ is entirely determined by the choices $\Phi^*_-(-), \Phi^-_*(-)$ of cartesian and cocartesian arrows, and the "filler maps" $\Phi^*_-(-;-), \Phi^-_*(-;-)$ are not necessary to the definition. But the notation we present for these is needed, as we will see.

It should be clear on how fibration structures are pulled back: given a type $(X, \underline{X})\colon \overline{X} \to A$ and an arbitrary map $f\colon B \to A$, we know that an object in the pullback $\overline{f^*X}$ is a pair (b, y) where b is an object of B and y is above fb in $\overline{X}$. Then given $m\colon a \to b$ and $n\colon b \to c$, calling Ψ the structure obtained by pulling back $\underline{X}$ by f, we take

$$m^*(b,y) \xrightarrow{\Psi^*_m(b,y)} (b,y) \quad \text{to be} \quad (a,(fm)^*y) \xrightarrow{(fm,\underline{X}^*_{fm}(y))} (b,y) \tag{5}$$

$$(b,y) \xrightarrow{\Psi^m_*(b,y)} n_*(b,y) \quad \text{to be} \quad (b,y) \xrightarrow{(fn,\underline{X}^{fn}_*(y))} (c,(fm)_*y)\ , \tag{6}$$

the same going for the rest of the structure.

The following result was noticed more than twenty years ago by the author, as a corollary of his thesis work [9]. It can also be deduced in the same manner from [17].

Theorem 1. *The class $\mathscr{S}$ of types as defined above has dependent products.*

Proof. We will only give the main features of the proof. Let $(X, \underline{X})\colon \overline{X} \to A$ and $(Y, \underline{Y})\colon \overline{Y} \to \overline{X}$ be two fibrations along with their defining structures, and let $(\Pi_X Y, \Phi)$ be the fibration and structure we want to describe. The Beck-Chevalley condition actually forces the definition of the functor $\Pi_X Y$. This is because, given a map m in A, and also calling $m\colon \mathbf{2} \to A$ the unique map it determines, the Beck-Chevalley condition forces the maps in $\overline{\Pi_X Y}$ above m to be in bijective correspondence with the "sections" ϕ

$$\begin{tikzcd}
 & \overline{Y} \arrow[d, "Y"] \\
\overline{m^*X} \arrow[ur, "\phi"] \arrow[r] \arrow[d, "m^*X"'] & \overline{X} \arrow[d, "X"] \\
\mathbf{2} \arrow[r, "m"'] & A
\end{tikzcd}$$

[2] We are aware that the notational tradition in geometry and topos theory makes $m_!x$ and $\Phi^m_!(x)$, etc. a more proper notation for the "cocartesian" half, but our choice of notation looks better and there is no risk of confusion in this paper.

that make the triangle above commute (the unnamed horizontal map is not necessarily an injection, so the term "section" is not entirely appropriate). The same being true, naturally, for objects, replacing **2** by **1** in that diagram. Thus, we define the objects and maps of $\overline{\Pi_X Y}$ as such "sections". We have defined a graph, but to make it a category, we have to use the (bi)fibration structure. Let ϕ and ψ be composable maps in $\overline{\Pi_X Y}$, with ϕ above $m\colon a \to b$ and ψ above $n\colon b \to c$. We know that $\psi\phi$ is a "section" $\overline{(nm)^* X} \longrightarrow \overline{Y}$, so let t be a map in $\overline{(nm)^* X}$. There are three possible cases that have to be examined, but let us assume that $t\colon x \to z$ is above nm (the other two are when t is above an identity, but it is rather obvious how to treat these). We take

$$(\psi\phi)(t) = \psi\big(\underline{X}_*^m(n;t)\big)\ \phi\big(\underline{X}_*^m(x)\big) = \psi(\underline{X}_n^*(z))\ \phi(\underline{X}_n^*(m;t))$$

the equality of these two values being guaranteed by the presence of the dotted arrow $\overline{X}_n^*(1_b; \underline{X}_*^m(n;t))$ in the diagram below.

$$\begin{tikzcd}
 & m_* x \arrow[rrd, "\underline{X}_*^m(n;t)"] \arrow[rdd, dotted] & & \\
x \arrow[ru, "\underline{X}_*^m(x)"] \arrow[rrr, "t"] \arrow[rrd, "\underline{X}_n^*(m;t)"'] & & & z \\
 & & n^* z \arrow[ru, "\underline{X}_n^*(z)"'] &
\end{tikzcd}$$

and the fact that $\underline{X}_*^m(x) \circ \overline{X}_n^*(1_b; \underline{X}_*^m(n;t)) = \underline{X}_n^*(m;t)$.

There is still a lot of things that need to be proved in order to show that $\overline{\Pi_X Y}$ is a category and $\Pi_X Y$ a functor, but it is rather mechanical. This category structure does not depend on the actual structures $\underline{X}, \underline{Y}$, but the structure Φ does. In order to show what the latter looks like, let us show how the choice of cocartesian map is done. So let $m\colon a \to b$ be a map in A and $\alpha\colon \overline{a^* X} \to \overline{Y}$ be a "section" above a. We want to define $\Phi_*^m(\alpha)\colon \alpha \to m_*\alpha$. First let us describe the codomain $m_*\alpha$ as a "section": given an object y in $\overline{X}$ above y we take

$(m_*\alpha)y = (\underline{X}^*_m(y))_*(\alpha(m^*y))$.

$$
\begin{tikzcd}
& \alpha(m^*y) \arrow[rr, "\underline{Y}_*^{\underline{X}^*_m(y)}(\alpha(m^*y))"] & & \big(\underline{X}^*_m(y)\big)_*(\alpha(m^*y)) \\
\alpha(x) \arrow[ur, "\alpha(\underline{X}^*_m(1_a;s))"] \arrow[urrr, "(\Phi^m_*(\alpha))s"'] & & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& m^*y \arrow[rr, two heads, "\underline{X}^*_m(y)"] & & y \\
x \arrow[ur, "\underline{X}^*_m(1_a;s)"] \arrow[urrr, two heads, "s"'] & & &
\end{tikzcd}
\tag{7}
$$

$$
\begin{tikzcd}
& a \arrow[rr, two heads, "m"] & & b \\
a \arrow[ur, equal] \arrow[urrr, two heads, "m"'] & & &
\end{tikzcd}
$$

The three horizontal arrows of the diagram above shows what is going on. It should be clear how this definition can be extended to compute $(m_*\alpha)r$ when r is a map in the fiber above b. In order to define $\Phi^m_*(\alpha)$ as a "section" let now s be a map above m, which is the missing one of three cases. The diagram shows the value of $\big(\Phi^m_*(\alpha)\big)s$ as the composite $\underline{Y}_*^{\underline{X}^*_m(y)}(\alpha(m^*y)) \circ \alpha(\overline{X}^*_m(1_a;s))$. There is still a lot to do to get a complete proof, in particular making the filler maps explicit, but it is all quite mechanical.

4. The Path Functor

We will first restrict ourselves to non-dependent types, by defining a path *endofunctor* $\mathbf{P}$ on *Cat*. The additional work required for the general case is comparatively a formality, as we will see. To avoid notational clashes, given $f\colon X \to Z$ in *Cat* will denote the action of $\mathbf{P}$ with parentheses: $\mathbf{P}(f)\colon \mathbf{P}(X) \to \mathbf{P}(Y)$. The functor $\mathbf{P}$ will actually be obtained by quotienting another endofunctor $\mathbf{P}$e on *Cat*.

So let X be a small category.

Definition 2. *An* elementary path $\mathbf{p}$ *in* X *is a quadruple*

$$\mathbf{p} \;=\; \big(\mathbb{I}, \leqslant, \sqsubseteq, (\mathbf{p}_{x,x'})_{x,x'}\big)$$

where

- $(\mathbb{I}, \leqslant)$ *is a nonempty finite totally ordered set, the* before-after order. *Thus when $x \leqslant y$ we say "x is before y, y after x", etc. We denote its first element by* b *(the beginning) and its last one by* e *(the end of the path).*

- $\sqsubseteq$ *is another order structure on* $\mathbb{I}$*, the* diagrammatic order, *that obey the following condition, in which* $\prec_{\leqslant}, \prec_{\sqsubseteq}$ *mean the predecessor relation on* $\leqslant, \sqsubseteq$ *respectively:*

$$\text{if } x \prec_{\leqslant} y \text{ then either } x \prec_{\sqsubseteq} y \text{ or } y \prec_{\sqsubseteq} x.$$

- $(\mathbf{p}_{x,x'})_{x,x' \in \mathbb{I}}$ *is a diagram* $(\mathbb{I}, \sqsubseteq) \to X$*. That is, for every* $x \in \mathbb{I}$ *there is an object* $\mathbf{p}_x \in X$ *and for every* $x \sqsubseteq x'$ *there is* $\mathbf{p}_{x,x'} \colon \mathbf{p}_x \to \mathbf{p}_{x'}$*, with the usual functorial identities. In particular* $\mathbf{p}_{x,x}$ *is the identity on the object* $\mathbf{p}_x$*.*

The *length* of an elementary path $\mathbf{p}$ is $\mathrm{Card}(\mathbb{I}_{\mathbf{p}}) - 1$.

When we deal with several elementary diagrams we use subscripts to distinguish whas has to be distinguished, e.g., $\mathbb{I}_{\mathbf{p}}, \sqsubseteq_{\mathbf{p}} \ldots$

Thus, if $\leqslant$ is a total order, we see that $\sqsubseteq$ has the shape of a zigzag, whose "branches" are totally ordered and coincide with segments of $\leqslant$, each branch of $\sqsubseteq$ having the induced order from $\leqslant$ or its opposite.

This definition is incomplete, because we need to identify two elementary paths $\mathbf{p}, \mathbf{q}$ that differ only by the way the elements or the indexing sets $\mathbb{I}_{\mathbf{p}}, \mathbb{I}_{\mathbf{q}}$ are named. Given two arbitrary elementary paths $\mathbf{p}, \mathbf{q}$, there is a natural definition of isomorphism between the structures (biposets) defined by the triples $\big(\mathbb{I}_{\mathbf{p}}, \leqslant, \sqsubseteq\big)$ and $\big(\mathbb{I}_{\mathbf{q}}, \leqslant, \sqsubseteq, \big)$: it's just a bijection between $\mathbb{I}_{\mathbf{p}}, \mathbb{I}_{\mathbf{q}}$ that preseves and reflects both orders. Since $\leqslant$ is a total finite order, if an iso $\alpha \colon \mathbb{I}_{\mathbf{p}} \to \mathbb{I}_{\mathbf{q}}$ exists it is unique, and we identify $\mathbf{p}, \mathbf{q}$ when we have $\mathbf{q}_{\alpha(x)} = \mathbf{p}_x$ for every $x \in \mathbb{I}_{\mathbf{p}}$. We could define the concept of elementary path without having to resort to quotienting by decreeing that $\mathbb{I}$ is always of the form $\{0, \ldots, n\}$ but this is more complicated because composing paths forces renamings.

Remark 1. The terminology before-after reminds one of a progress in time, which is a pretty traditional way of thinking of paths in homotopy theory as in geometry. But a category-theoretical tradition also would like us to call this order the *vertical order.*

Here is an elementary path of length 6, drawn vertically.

$$\begin{tikzcd}[row sep=small]
\mathbf{p}_{\mathsf{b}} = \mathbf{p}_0 \arrow[d, "\mathbf{p}_{0,1}"'] \\
\mathbf{p}_1 \arrow[d, "\mathbf{p}_{1,2}"'] \\
\mathbf{p}_2 \\
\mathbf{p}_3 \arrow[u, "\mathbf{p}_{3,2}"] \\
\mathbf{p}_4 \arrow[u, "\mathbf{p}_{4,3}"] \\
\mathbf{p}_5 \arrow[u, "\mathbf{p}_{5,4}"] \arrow[d, "\mathbf{p}_{5,6}"'] \\
\mathbf{p}_{\mathsf{e}} = \mathbf{p}_6
\end{tikzcd} \tag{8}$$

The $\leqslant$ order is read from the top down, and so a down-arrow means that $\leqslant, \sqsubseteq$ coincide and an up-arrow the opposite. An elementary path of length zero is just an object of X; such a path is different from all the paths $\mathbf{p}$ all whose $\mathbf{p}_{x,y}$ are identities, which can have arbitrary length $n \geqslant 1$

Definition 3. *Let $(\mathbb{I}, \sqsubseteq), (\mathbb{J}, \sqsubseteq)$ be two posets.* An ordering from $\mathbb{I}$ to $\mathbb{J}$ *is an ordering $\sqsubseteq$ of the disjoint sum $\mathbb{I} + \mathbb{J}$ such that*

- *the restriction of $\sqsubseteq$ on $\mathbb{I}, \mathbb{J}$ is exactly $\sqsubseteq_{\mathbb{I}}, \sqsubseteq_{\mathbb{J}}$.*
- *if $x \in \mathbb{I}, y \in \mathbb{J}$ are related by the $\sqsubseteq$ order, then $x \sqsubseteq y$.*

Proposition 1. *Let $\mathbb{I}, \mathbb{J}$ be as above and $\sqsubseteq$ an ordering from $\mathbb{I}$ to $\mathbb{J}$. By restriction this order determines a relation $R \subseteq \mathbb{I} \times \mathbb{J}$, i.e., xRy iff $x \sqsubseteq y$. This restriction operation determines a bijection between the set of orderings from $\mathbb{I}$ to $\mathbb{J}$ and the set of relations $R \subseteq \mathbb{I} \times \mathbb{J}$ that are*

- *left-down-closed: $xRy, x' \sqsubseteq x$ implies $x'Ry$,*
- *right-up-closed: $xRy, y' \sqsupseteq y$ implies xRy'.*

Proof. Very easy.

Now it is well known that the class of posets along with left-down- and right-up-closed relations form a category, which has been dubbed by Lambek the category of posets and comparisons [11] and is a very special case of the extremely general bimodule/profunctor/distributor construction in enriched category theory [12].

In other words, given comparisons $R \subseteq \mathbb{I} \times \mathbb{J}$ and $S \subseteq \mathbb{J} \times \mathbb{K}$ it is easy to check that

$$S \circ R \quad = \quad \{\, (x, z) \in \mathbb{I} \times \mathbb{K} \mid \text{there exists } y \in \mathbb{J} \text{ with } (x, y) \in R, (y, z) \in S \,\}$$

is a comparison too, and that for any poset $\mathbb{I}$ the left-down- and right-up-closure of the identity relation acts as the identity for composition (but not the identity relation itself, since it is not a comparison unless $\mathbb{I}$ is a discrete poset!).

Definition 4. *Given elementary paths* $\mathbf{p}, \mathbf{q}$ *in* X *we define a* premap $\mathbf{f} \colon \mathbf{p} \to \mathbf{q}$ *to be a diagram* $\mathbf{f} \colon (\mathbb{I}_\mathbf{p} + \mathbb{I}_\mathbf{q}, \sqsubseteq_\mathbf{f}) \to X$, *where* $\sqsubseteq_\mathbf{f}$ *is an order from* $(\mathbb{I}_\mathbf{p}, \sqsubseteq)$ *to* $(\mathbb{I}_\mathbf{q}, \sqsubseteq)$, *that obeys the additional conditions*

1. $\mathbf{p}_\mathsf{b} \sqsubseteq \mathbf{q}_\mathsf{b}$ *and* $\mathbf{p}_\mathsf{e} \sqsubseteq \mathbf{q}_\mathsf{e}$
2. $\mathbf{f}$ *restricted to* $\mathbb{I}_\mathbf{p}, \mathbb{I}_\mathbf{q}$ *is* $\mathbf{p}, \mathbf{q}$.

The poset $(\mathbb{I}_\mathbf{p} + \mathbb{I}_\mathbf{q}, \sqsubseteq_\mathbf{f})$ *is called the* shape *of* $\mathbf{f}$. *A premap* $\mathbf{f}$ *is said to be a* map, *when it obeys in addition the following* contractibility conditions:

a) *given* $x_1, x_2, x \in \mathbb{I}_\mathbf{p}$ *where* x *is* $\leqslant$*-between* x_1, x_2, *along with* $y_1 \sqsupseteq x_1, y_2 \sqsupseteq x_2$ *in* $\mathbb{I}_\mathbf{q}$, *then there exists* y $\leqslant$*-between* y_1, y_2 *with* $y \sqsupseteq x$.
b) *given* $y_1, y_2, y \in \mathbb{I}_\mathbf{q}$ *where* y *is* $\leqslant$*-between* y_1, y_2, *along with* $x_1 \sqsubseteq y_1, x_2 \sqsubseteq y_2$ *in* $\mathbb{I}_\mathbf{p}$, *then there exists* x $\leqslant$*-between* x_1, x_2 *with* $x \sqsubseteq y$.

Using Proposition 1 we can get a slightly different definition of map or premap between elementary paths: a premap $\mathbf{f} \colon \mathbf{p} \to \mathbf{q}$ can be seen as a pair $(G, \mathbf{f})$ where $G \subseteq \mathbb{I}_\mathbf{p} \times \mathbb{I}_\mathbf{q}$ (the *graph* of $\mathbf{f}$), where $\mathbf{f}_{x,y}$ is a map $\mathbf{p}_x \to \mathbf{q}_y$ for every $(x, y) \in G$, Condition 1 can be rephrased as

- (b, b) and (e, e) are in G,

and Condition 2 along with functoriality of $\mathbf{f}$ can be rephrased as

- Given $x, x' \in \mathbf{p}$ and $y, y' \in \mathbf{q}$ such that $x' \sqsubseteq x \sqsubseteq y \sqsubseteq y'$ then $\mathbf{q}_{y,y'} \circ \mathbf{f}_{x,y} \circ \mathbf{p}_{x',x} = \mathbf{f}_{x',y'}$.

In practice we can make things even simpler and just define a map to be the family $(\mathbf{f}_{x,y})_{x,y}$ since G can be deduced as the set of (x, y) such that $f_{x,y}$ is defined. Then for example the first contractibility condition can be rephrased as

- given $\mathbf{f}_{x_1,y_1}$, $\mathbf{f}_{x_2,y_2}$ and x which is $\leqslant$-between x_1, x_2 then there is $\mathbf{f}_{x,y}$ where y is $\leqslant$-between y_1, y_2.

Proposition 2. *Let* $\mathbb{I}, \mathbb{J}, \mathbb{K}$ *be posets and* $R \colon \mathbb{I} \to \mathbb{J}, S \colon \mathbb{J} \to \mathbb{K}$ *be comparisons that both satisfy Conditions 1,2 above (as posets). Then* $S \circ R$ *also obeys these conditions. This is also true if Conditions a,b hold additionally.*

Proof. The proof is trivial.

Given maps $\mathbf{f} \colon \mathbf{p} \to \mathbf{q}$ and $\mathbf{g} \colon \mathbf{q} \to \mathbf{r}$ we define the *horizontal composite*, written $\mathbf{gf}$ or $\mathbf{g} \circ \mathbf{f}$, as the family

$$(\mathbf{g}_{yz} \circ \mathbf{f}_{xy})_{xz} \mid \text{there exists } y \text{ with both } \mathbf{g}_{yz} \text{ and } \mathbf{f}_{xy} \text{ defined.}$$

But we first have to show this is actually a real definition, i.e., ensure that we always have $g_{yz} \circ f_{xy} = g_{y'z} \circ f_{xy'}$. So let $x \in \mathbb{I}_{\mathbf{p}}, y, y' \in \mathbb{I}_{\mathbf{q}}, z \in \mathbb{I}_{\mathbf{r}}$ be such that the situation above happens. Let $y_0 = y, y_1, \dots y_{n-1}, y_n = y'$ be a sequence of elements of $\mathbb{I}_{\mathbf{q}}$ that are all $\leqslant$-between y, y' and such that we always have $y_{i-1} \sqsubseteq y_i \sqsupseteq y_{i+1}$ or $y_{i-1} \sqsupseteq y_y \sqsubseteq y_{i+1}$. In other words we take all the "$\sqsubseteq$-peaks" and "$\sqsubseteq$-valleys" between y, y', respecting the $\leqslant$-order. If we choose an arbitrary $0 \leqslant i \leqslant n$ we know because of (the equivalent of) Condition 3 that f_{x,y_i} is guaranteed to exist, and because of Condition 2 that $g_{y_i,z}$ is guaranteed to exist. But the fact that y_i, y_{i+1} are $\sqsubseteq$-related guarantees that $g_{y_i z} \circ f_{xy_i} = g_{y_{i+1} z} \circ f_{xy_{i+1}}$ and this shows $g_{yz} \circ f_{xy} = g_{y'z} \circ f_{xy'}$ by induction.

Thus we get

Proposition 3. *Given a small category X the set of elementary paths and maps in X, along with horizontal composition, form a category $\mathbf{Pe}(X)$.*

The identity on a path $\mathbf{p}$ is obtained by taking the family $(\mathbf{p}_{x,x'})_{x \sqsubseteq x'}$ and closing under pre-post-composition.

Actually, it is more than just a category, but an order-enriched one.

Definition 5. *Let $\mathbf{f}, \mathbf{g} \colon \mathbf{p} \to \mathbf{q}$ be maps. We write $\mathbf{f} \subseteq \mathbf{g}$ if $\mathbf{g}_{x,y}$ is defined whenever $\mathbf{f}_{x,y}$ is, and $\mathbf{g}_{x,y} = \mathbf{f}_{x,y}$.*

Trivially, this is an order relation, and the operation of horizontal composition is monotone in both variables.

We also have obvious functors $\partial_0, \partial_1 \colon \mathbf{Pe}(X) \to X$ that take a map of paths $\mathbf{f} \colon \mathbf{p} \to \mathbf{q}$ and restrict it to its endpoints, getting $\mathbf{f}_{\mathsf{b}} \colon \mathbf{p}_{\mathsf{b}} \to \mathbf{q}_{\mathsf{b}}$ and $\mathbf{f}_{\mathsf{e}} \colon \mathbf{p}_{\mathsf{e}} \to \mathbf{q}_{\mathsf{e}}$ respectively. There is also an obvious $\mathsf{r}X \colon X \to \mathbf{Pe}X$ which is a section to both ∂_0, ∂_1, that takes an object to the path of length zero it defines, composed of that object alone.

Proposition 4. *The process $\mathbf{Pe}(-)$ defines an endofunctor on Cat and $\partial_0, \partial_1, \mathsf{r}$ are natural.*

Proof. Given a map of categories $f \colon X \to Y$, any morphism $\mathbf{f} \colon \mathbf{p} \to \mathbf{q}$ in $\mathbf{Pe}(X)$ can be seen as a diagram $\mathbf{f} \colon (\mathbb{I}_{\mathbf{p}} + \mathbb{I}_{\mathbf{q}}, \sqsubseteq_{\mathbf{f}}) \to X$ and thus $f \circ \mathbf{f}$ will be a morphism in $\mathbf{Pe}(Y)$. Functoriality of the assignment $\mathbf{f} \mapsto f \circ \mathbf{f}$ is trivial to obtain, as is the naturality of $\partial_0, \partial_1, \mathsf{r}$.

4.1. Vertical composition

Let $\mathbf{p}, \mathbf{p}'$ be two paths such that $\mathbf{p}_{\mathsf{e}} = \mathbf{p}'_{\mathsf{b}}$. It is quite obvious how to concatenate the two to get the vertical composite $\mathbf{p}' * \mathbf{p}$. This is just a pushout construction that involves the two orders $\leqslant, \sqsubseteq$. This also applies to maps of paths: given $\mathbf{f} \colon \mathbf{p} \to \mathbf{q}$ and $\mathbf{f}' \colon \mathbf{p}' \to \mathbf{q}'$ with $\mathbf{f}_{\mathsf{e}} = \mathbf{f}'_{\mathsf{b}}$ a slightly more involved pushout constructs

$$\mathbf{f}' * \mathbf{f} \colon \mathbf{p}' * \mathbf{p} \longrightarrow \mathbf{q}' * \mathbf{q}.$$

Proposition 5. *We always have*

$$(\mathbf{k} \circ \mathbf{h}) * (\mathbf{g} \circ \mathbf{f}) \;\subseteq\; (\mathbf{k} * \mathbf{g}) \circ (\mathbf{h} * \mathbf{f})$$

whenever $\mathbf{f}, \mathbf{g}, \mathbf{h}, \mathbf{k}$ *are four maps of path for which the above is defined.*

PROOF. Easy.

This is not an equation, and here is a counterexample:

$$\begin{tikzcd}
\cdot \arrow[d] \arrow[dr, "\mathbf{f}"'] & & \cdot \arrow[d] \\
\cdot \arrow[d] \arrow[r] & \cdot \arrow[ur, "\mathbf{h}"'] \arrow[r] \arrow[dr, "\mathbf{k}"] & \cdot \arrow[d] \\
\cdot \arrow[ur, "\mathbf{g}"] & & \cdot
\end{tikzcd} \tag{9}$$

It is easy to see that in $(\mathbf{k} * \mathbf{g}) \circ (\mathbf{h} * \mathbf{f})$ there is an arrow from bottom left to top right of the resulting square which is not in $(\mathbf{k} \circ \mathbf{h}) * (\mathbf{g} \circ \mathbf{f})$.

Thus vertical composition is not a (bi)functorial operation. It is only colax-functorial when we consider the $\subseteq$ enrichment. Nonetheless it is associative on the nose and has a real unit, and can be considered as a functorial binary operation over the category of graphs instead of Cat.

We need a notation for triangles like those we've just used. First, given a map s in a category X we write $[s]$ for the path of length 1 that has this unique map, pointing in its natural direction (downwards) and $[\![s]\!]$ for its reverse, where s points upwards. Then the four "elementary triangles" are as follows:

$$\begin{array}{cccc}
\begin{tikzcd}
\cdot \arrow[d, "r"'] \arrow[dr, "sr"] & \\
\cdot \arrow[r, "s"'] & \cdot
\end{tikzcd}
&
\begin{tikzcd}
\cdot \arrow[dr, "s"] & \\
\cdot \arrow[u, "r"] \arrow[r, "sr"'] & \cdot
\end{tikzcd}
&
\begin{tikzcd}
& \cdot \arrow[d, "r"] \\
\cdot \arrow[ur, "s"] \arrow[r, "rs"'] & \cdot
\end{tikzcd}
&
\begin{tikzcd}
& \cdot \\
\cdot \arrow[ur, "rs"] \arrow[r, "s"'] & \cdot \arrow[u, "r"']
\end{tikzcd}
\\
[r, s\rangle & [\![r, s\rangle & \langle s, r] & \langle s, r]\!]
\end{array} \tag{10}$$

This notation should be easy to memorize. Finally given an object x we write $\langle x\rangle$ for the one-object path it defines, of length zero, and given $s\colon x \to y$ in X we take $\langle s\rangle\colon \langle x\rangle \to \langle y\rangle$ to be the corresponding map of paths. Thus the operation $\langle -\rangle$ is $\mathsf{r}X$ and furnishes the identities for vertical composition.

These four triangles, along with the vertical identities, generate the whole of $\mathbf{Re}(X)$, in a sense which has to made explicit and is rather nontrivial. A special and important case of this is how horizontal identities are obtained. Let $s\colon x \to y$ be a map in X. Then the identity maps of $[s]$, $[\![s]\!]$ are

$$[s, 1_y\rangle * \langle 1_x, s] \quad \text{and} \quad \langle 1_x, s]\!] * [\![s, 1_y\rangle$$

respectively.

Remark 2. In what follows we will write $[s]$, $[\![s]\!]$ for these two identity maps. In other words we will use the objects-are-identity-maps definitional paradigm when dealing with horizontal composition, which simplifies notation considerably.

Let us show that the functor $\langle \partial_0, \partial_1 \rangle \colon \mathbf{Pe}(X) \to X \times X$ has very fibration-like properties. First given $(x, x') \in X \times X$, let us say that $\mathbf{p}$ *is above* (x, x') to mean that a path $\mathbf{p}$ is such that $\mathbf{p}_{\mathsf{b}} = x, \mathbf{p}_{\mathsf{e}} = x'$. Given $(s, s') \colon (x, x') \to (y, y')$ and $\mathbf{p}$ above (x, x') we define

$$
\begin{aligned}
(s, s')_* \mathbf{p} &= \left[s'\right] * \mathbf{p} * [\![s]\!] && (11)\\
\underline{\mathbf{Pe}X}_*^{(s,s')}(\mathbf{p}) &= [1_{x'}, s'\rangle * 1_{\mathbf{p}} * [\![1_x, s\rangle \ : \mathbf{p} \longrightarrow (s, s')_* \mathbf{p} && (12)
\end{aligned}
$$

and given $\mathbf{q}$ above (y, y') we define

$$
\begin{aligned}
(s, s')^* \mathbf{q} &= [\![s']\!] * \mathbf{q} * [s] && (13)\\
\underline{\mathbf{Pe}X}^*_{(s,s')}(\mathbf{q}) &= \langle 1_{y'}, s']\!] * 1_{\mathbf{q}} * \langle 1_y, s] \ : (s, s')^* \mathbf{q} \longrightarrow \mathbf{q} \,. && (14)
\end{aligned}
$$

These maps do act a lot like (co)cartesians. For instance:

Proposition 6. *Let* $\mathbf{p}$ *and* (s, s') *be as above, and let* $(r, r') \colon (y, y') \to (z, z')$ *be in* X *and* $\mathbf{f}$ *be above* $(rs, r's')$*. Then*

$$
[s', r'\rangle * \mathbf{f} * [\![s, r\rangle \ \circ \ \underline{\mathbf{Pe}X}_*^{(s,s')}(\mathbf{p}) \quad = \quad \mathbf{f} \,.
$$

Thus we can define $\underline{\mathbf{Pe}X}_*^{(s,s')}((r, r'); \mathbf{f}) = [s', r'\rangle * \mathbf{f} * [\![s, r\rangle$. But this "filler" map is not uniquely defined in general. For a counterexample, suppose that s, s' are isos. Then

$$
\left(\langle s^{-1}, s] * [s, s^{-1}\rangle * 1_{\mathbf{p}} * [\![s, s^{-1}\rangle * \langle s^{-1}, s]\!]\right) \ \circ \ \underline{\mathbf{Pe}X}_*^{(s,s')}(\mathbf{p}) \quad = \quad \underline{\mathbf{Pe}X}_*^{(s,s')}(\mathbf{p}) \,.
$$

But $\langle s^{-1}, s] * [s, s^{-1}\rangle * 1_{\mathbf{p}} * [\![s, s^{-1}\rangle * \langle s^{-1}, s]\!]$ is not the identity (which is also $\underline{\mathbf{Pe}X}_*^{(s,s')}((r, r'); \mathbf{f})$) because it contains two extra maps; this is very easy to see. Thus $\underline{\mathbf{Pe}X}_*^{(s,s')}(\mathbf{p})$ is not a real cocartesian in general and we do not get a standard bifibration, but something a little more general. Characterizing these generalized fibrations is a question that reaches beyond the scope of the present work.

Definition 6. *We define* $\mathbf{P}(X)$ *to be the quotient of* $\mathbf{Pe}(X)$ *under the symmetric closure of the order enrichment* $\subseteq$*. There is an obvious functor* $\mathbf{Pe}(X) \to \mathbf{P}(X)$ *and obvious projection* $\mathbf{P}(X) \to X \times X$ *making the obvious triangle commute.*

The objects of $\mathbf{P}(X)$ coincide with those of $\mathbf{Pe}(X)$, they are elementary paths. The maps are equivalence classes of maps of elementary maps. These classes can be quite complex, and there is no hope of finding normal forms in general. But notice that the class of a triangle is always a singleton, as is the class of a vertical unit. Notice also that Inequality (5) is now an equation in the quotient, and thus vertical composition is $\mathbf{P}(X)$ is bifunctorial. Thus $\mathbf{P}(X)$ is an object of maps for an internal category whose object of objects is X. We have constructed a natural double category structure on every small category.

Definition 7. *Let* $\mathbf{Q}$ *stand for one of* $\mathbf{Pe}, \mathbf{P}$ *and* $Y \colon \overline{Y} \to X$ *be in* Cat. *We define a* Hurewicz $\mathbf{Q}$-action on Y *to be a splitting* ρ *of the map* $K \colon \mathbf{Q}(\overline{Y}) \to \overline{\partial_0^* Y}$ *determined by the universal property of pullback.*

$$
\begin{tikzcd}
 & \mathbf{Q}\,(\overline{Y}) \arrow[ddl, "\partial_0"'] \arrow[d, "K"'] \arrow[ddr, "\mathbf{Q}\,(Y)"] & \\
 & \overline{\partial_0^* Y} \arrow[u, bend right, "\rho"'] \arrow[dl] \arrow[dr, "\partial_0^* Y"'] & \\
\overline{Y} \arrow[dr, "Y"'] & & \mathbf{Q}\,(X) \arrow[dl, "\partial_0"] \\
 & X &
\end{tikzcd}
$$

Thus a map in $\overline{\partial_0^* Y}$ *is a pair* $(\mathbf{f}, s)$ *where* $\mathbf{f}$ *is a map in* $\mathbf{Q}\,(X)$, *(either an map of elementary paths in* X *or a set of such parallel maps) and* $s \in \overline{Y}$ *is above* $\partial_0 X(\mathbf{f})$. *Let us write* $\rho(\mathbf{f}, s)$ *as* $\mathbf{f} \star s$. *In addition to being a functor,* ρ *is required to respect vertical composition in the following sense: for every three maps* $\mathbf{f}, \mathbf{g}$ *in* $\mathbf{Q}(X)$, s *in* $\overline{Y}$ *such that* $\mathbf{g} * \mathbf{f}$ *and* $\mathbf{f} \star s$ *are defined, we have*

$$
(\mathbf{g} * \mathbf{f}) \star s \;=\; \big(\mathbf{g} \star \partial_1(\mathbf{f} \star s)\big) * (\mathbf{f} \star s)\ . \tag{15}
$$

It is not hard to see that every splitting ρ of K automatically preserves the vertical unit, in other words that for very m in X and $s \in \overline{Y}$ above m, we have

$$
\langle m \rangle \star s \;=\; \langle s \rangle\ . \tag{16}
$$

This is because the vertical unit is the only map of paths of length zero, and thus it is necessarily preserved by K, this being true for both possible choices of $\mathbf{Q}$.

Let us for a moment assume that $\mathbf{Q} = \mathbf{Pe}$. Given $\mathbf{f}$ and s such that $\mathbf{f} \star s$ is defined, we know that $\mathbf{f} = \mathbf{Q}\,(Y)(\mathbf{f} \star s)$, and since $\mathbf{Q}\,(Y)$ preserves shapes, $\mathbf{f} \star s$ has exactly the same shape as s. Also, since horizontal identity maps in $\overline{\partial_0^* Y}$ are pairs of the form $([m]\,, 1_x)$, the path $[m] \star 1_x$ is a horizontal identity, i.e., it is of the form $[r]$ for a map r in $\overline{Y}$. Let us denote this map by

$$
\Phi\rho_*^m(x) \colon\; x \longrightarrow m_* x \quad \text{i.e.,} \quad [m] \star 1_x \;=\; [\Phi\rho_*^m(x)]\ . \tag{17}
$$

Assuming that $m \colon a \to b$, we know that

$$
[m] = [m, 1_b\rangle * \langle 1_a, m] \text{ and } [\Phi\rho_*^m(x)] = [\Phi\rho_*^m(x), 1_{m_* x}\rangle * \langle 1_x, \Phi\rho_*^m(x)]
$$

and plugging these back in Equation (17)

$$
\big([m, 1_b\rangle * \langle m, 1_a]\big) \star 1_x \;=\; [\Phi\rho_*^m(x), 1_{m_* x}\rangle * \langle 1_x, \Phi\rho_*^m(x)] \tag{18}
$$

but Equation (15) also tells us

$$\big([1_b, m\rangle * \langle m, 1_a]\big) \star 1_x \;=\; \big([m, 1_b\rangle \star \partial_1\big(\langle 1_a, m] \star 1_x\big)\big) * \big(\langle 1_a, m] \star 1_x\big)\ . \tag{19}$$

Due to the unique way of decomposing the shape of such paths into triangles, we can deduce from the equality between the right sides of Equations (18,19) that

$$\langle 1_a, m] \star 1_x \;=\; \langle 1_x, \Phi\rho^m_*(x)] \qquad \text{and} \tag{20}$$

$$[m, 1_b\rangle \star \partial_1\big(\langle 1_a, m] \star 1_x\big) \;=\; [\Phi\rho^m_*(x), 1_{m_*x}\rangle \tag{21}$$

and since $\partial_1\big(\langle 1_a, m] \star 1_x\big) \;=\; \partial_1\langle 1_x, \Phi\rho^m_*(x)] \;=\; \Phi\rho^m_*(x)$ we can rewrite Equation (21) as

$$[m, 1_b\rangle \star \Phi\rho^m_*(x) \;=\; [\Phi\rho^m_*(x), 1_{m_*x}\rangle\ . \tag{22}$$

Let now $n\colon b \to c$ be in X and $r\colon x \to z$ be above nm. The elementary path $[m, n\rangle \star r$ is a triangle, so let $t\colon x \to y$ and $\Phi\rho^m_*(n; r)\colon y \to z$ be the maps that define this triangle, i.e.,

$$[m, n\rangle \star r \;=\; [t, \Phi\rho^m_*(n; r)\rangle\ .$$

Since $[m, n\rangle$ is a map $[m] \to \langle c\rangle$ in $\mathbf{Pe}(X)$ and ρ is functorial, we have

$$[m] \star 1_x = [\Phi\rho^m_*(x)] \xrightarrow{[m,n\rangle \star r} \langle z\rangle$$

and this shows

$$t = \Phi\rho^m_*(x) \text{ and } \Phi\rho^m_*(n; r) \circ \Phi\rho^m_*(x) = r\ .$$

Now take $u\colon m_*x \to z$ any map such that $u \circ \Phi\rho^m_*(x) = r$. Obviously $\langle n\rangle \circ [m, 1_b\rangle = [m, n\rangle$. By functoriality of the action ρ, we also have

$$\begin{tikzcd}
 & x \arrow[r, "\Phi\rho^m_*(x)"] \arrow[dl, "\Phi\rho^m_*(x)"'] \arrow[dr, phantom, "{[m,1_b\rangle\star\Phi\rho^m_*(x)}"] & m_*x \arrow[r, "u", "\langle n\rangle \star u"'] & z \\
m_*x \arrow[urr, "1"'] & & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & a \arrow[r, "m"] \arrow[dl, "m"'] \arrow[dr, phantom, "{[m,1_b\rangle}"] & b \arrow[r, "n", "\langle n\rangle"'] & c \\
b \arrow[urr, "1"'] & & &
\end{tikzcd}$$

$$\big(\langle n\rangle \star u\big) \circ \big([m, 1_b\rangle \star \Phi\rho^m_*(x)\big) \;=\; \big(\langle n\rangle \circ [m, 1_b\rangle\big) \star \big(u \circ \Phi\rho^m_*(x)\big) \;=\; [m, n\rangle \star r$$

But by Equation (22) the left side of this is $\langle u\rangle \circ [\Phi\rho^m_*(x), 1_{m_*x}\rangle$ which is $[\Phi\rho^m_*(x), u\rangle$ trivially and we have seen above that the right side is $[\Phi\rho^m_*(x), \Phi\rho^m_*(n; r)\rangle$. Thus $u = \Phi\rho^m_*(n; r)$ and we have proved that $\Phi\rho^m_*(x)$ is a cocartesian arrow.

This can be done dually; that is, if $m\colon a \to b$ is in X and y in $\overline{Y}$ above b, then defining $\Phi\rho^*_m(y)\colon m^*y \to y$ as the unique s such that $[\![s]\!] = [\![m]\!] \star 1_y$, we can show that $\Phi\rho^*_m(y)\colon m^*y \to y$ is a Cartesian arrow.

Let now $\mathbf{Q} = \mathbf{P}$. The argument above only needs slight modifications to be applicable. First, given an arbitrary small category Z, recall that the equivalence class in $\mathbf{P}(Z)$ of a triangle or a path of length zero is a singleton, and notice that the only elementary paths in this argument that do not belong to the above are those of the form $[u]$ or $[\![u]\!]$ for a morphism u of Z. But it is easy to see that such a horizontal unit in $\mathbf{P}(Z)$ is either the singleton $\{[u]\}, \{[\![u]\!]\}$ or a doubleton, the latter case happening only if u is an isomorphism. Thus it is easy to see everything we have said also holds in that context.

So we conclude

Proposition 7. *Let $Y : \overline{Y} \to X$ be a functor and $\mathbf{Q}$ be either $\mathbf{Pe}$ or $\mathbf{P}$. Every Hurewicz $\mathbf{Q}$-action ρ on Y determines a bifibration structure $\Phi\rho$ on Y.*

This has a converse

Theorem 2. *The correspondence $\rho \mapsto \Phi\rho$ is bijective.*

An immediate consequence is that there is also a bijective correspondence between Hurewicz $\mathbf{Pe}$-actions and Hurewicz $\mathbf{P}$-actions on Y.

Theorem 3. *Given a small category X then $\mathbf{P}(X) \to X \times X$ is a bifibration.*

Thus, the "almost cartesian" and "almost cocartesian" maps in $\mathbf{Pe}(X)$ become truly cartesian and cocartesian in the quotient $\mathbf{P}(X)$.

The proofs of these results [10] is rather more technical and depends on a close analysis of the order enrichment $\subseteq$ on $\mathbf{Pe}$.

Remark 3. Our functor $\mathbf{P}$ comes very close to fitting the axiomatic framework of a *path object category* presented in [5], but there are real differences. First, the functor $\mathbf{P}$ doesn't preserve all pullbacks, as is required there—it doesn't even preseve all monos. Preservation of all pullbacks is an essential, foundational requirement in that work. It seems that the pullback squares whose preservation by the path functor is really necessary in general are those that have two parallel fibrations, and this holds in our model. Thus one aspect of our work seems to require a more general axiomatic treatment than the one presented in [5]. On the other hand, the construction therein that gives a fibration structure to an arbitrary map, making it a (cloven) $\mathscr{R}$-map in that paper's terminology, corresponds exactly in our model to a Hurewicz action where vertical composition—Equation (15)—is not necessarily preserved, and thus that framework is more general than ours in that respect. These maps are called weak Hurewicz actions in [10], and correspond exactly to Hurewicz's original definition. Our Theorem 2 tells us that Hurewicz actions as defined in the present paper correspond to a very well known concept in category theory, but we still do not know what weak Hurewicz actions are, and they might very well give another model for identity types (although we are almost certain that taking maps equipped with a weak Hurewicz action for abstract fibrations in a display category will not give a model with dependent products).

Obviously an abstract framework that would unify our work with that of [5] would be desirable, and lead to a better understanding of these kinds of models.

4.2. The fibered path functor

We will now briefly describe how the structure $(\mathbf{P}(-), \partial_0, \partial_1, \mathsf{r})$ can be extended to the fibered version that's defined at the beginning of Section 2.1. The following subsection will introduce the last missing ingredient, the J operator.

An elementary path is said to be *constant* if all its component maps are identities, or if it has length zero. This obviously defines a full subfunctor $\mathbf{Ce}$ for which the inclusion map $\mathsf{i}\colon \mathbf{Ce} \to \mathbf{Pe}$ obviously equalizes the two projections ∂_0, ∂_1, and we we call the composite ∂. By definition the transformation r factors through that inclusion:

$$
\begin{tikzcd}
\mathbf{Ce}(X) \arrow[rr, "\mathsf{i}"] \arrow[dr, "\partial"'] & & \mathbf{Pe}(X) \arrow[dl, shift right, "\partial_0"'] \arrow[dl, shift left, "\partial_1"] \\
& X &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{Ce}(X) \arrow[rr, "\mathsf{i}"] & & \mathbf{Pe}(X) \\
& X \arrow[ul, "\mathsf{r}"] \arrow[ur, "\mathsf{r}"'] &
\end{tikzcd}
$$

Notice that given a category X and constant paths $\mathbf{p}, \mathbf{q}$, then taking a map $\mathbf{f}\colon \mathbf{p} \to \mathbf{q}$ we see that all its components $\mathbf{f}_{x,y}$ are the same map of X. Moreover every such $\mathbf{f}$ has an $\subseteq$-maximal element above it, whose shape is the full product $\mathbb{I}_{\mathbf{p}} \times \mathbb{I}_{\mathbf{q}}$. Thus when we take the associated quotient $\mathbf{Ce}(X) \to \mathbf{C}(X)$ not only do we also get a subfunctor of $\mathbf{P}$, whose inclusion (also called i) equalizes ∂_0, ∂_1, but in addition the composite ∂ is an equivalence between $\mathbf{C}(X)$ and X.

$$
\begin{tikzcd}
\mathbf{Ce}(X) \arrow[r, two heads] \arrow[d, "\mathsf{i}"'] & \mathbf{C}(X) \arrow[d, "\mathsf{i}"'] \arrow[dr, "\partial"] & \\
\mathbf{Pe}(X) \arrow[r, two heads] & \mathbf{P}(X) \arrow[r, shift left, "\partial_0 X"] \arrow[r, shift right, "\partial_1 X"'] & X
\end{tikzcd}
$$

Let now $X\colon \overline{X} \to A$ be a bifibration. In order to define $\mathbf{Pe}X\colon \overline{\mathbf{Pe}X} \to X \times_A X$ we take $\overline{\mathbf{Pe}X} \subseteq \mathbf{Pe}(\overline{X})$ to be the full subcategory of all elementary paths in $\overline{X}$ that are mapped by X to a constant path in A. In other words we obtain $\overline{\mathbf{Pe}X}$ by pulling back $\mathbf{Pe}(X)$ by i. This ensures that the inclusion map followed by the projection $\langle \partial_0 X, \partial_1 X \rangle$ will factor through $\overline{X \times_A X}$ and this defines the map of categories $\mathbf{Re}X$.

$$\begin{tikzcd}
\overline{X \times_A X} \arrow[rrr] \arrow[ddd, "X \times_A X"'] & & & \overline{X} \times \overline{X} \arrow[ddd, "X \times X"] \\
& \overline{\mathbf{Re}X} \arrow[ul, "\mathbf{Re}X"] \arrow[r] \arrow[d, "\mathsf{i}^*\mathbf{Re}(X)"'] & \mathbf{Re}(\overline{X}) \arrow[ur, "\langle \partial_0 X, \partial_1 X \rangle"] \arrow[d, "\mathbf{Re}(X)"] & \\
& \mathbf{Ce}(A) \arrow[r, "\mathsf{i}"'] \arrow[dl, "\partial"] & \mathbf{Re}(A) \arrow[dr, "\langle \partial_0 A, \partial_1 A \rangle"'] & \\
A \arrow[rrr, "\Delta"'] & & & A \times A
\end{tikzcd}$$

Now if Φ is the cleavage structure for the almost-a-Grothendieck-bifibration $\langle \partial_0 X, \partial_1 X \rangle$, as given in Equations (11–14), it is easy to see that the operations $\Phi^*_-(-), \Phi^-_*(-)$, etc. when restricted to the subfunctor $\mathbf{Re}X$, will define another such cleavage structure Ψ on $\mathbf{Re}X$.

This definition of $\mathbf{Re}X$ says that it really is "the original $\mathbf{Re}(-)$ on Cat applied fiberwise" In particular it is easy to see that the order enrichment $\subseteq$ is fiberwise, i.e., if two parallel maps in $\overline{\mathbf{Re}X}$ are $\subseteq$-related then they are above the same map in A. Thus, if we define the category $\overline{\mathbf{P}X}$ by quotienting with that order, $\mathbf{Re}X$ will factor through that quotient, making the expected triangle commute.

$$\begin{tikzcd}
\overline{\mathbf{Re}X} \arrow[rr, two heads] \arrow[dr, "\mathbf{Re}X"'] & & \overline{\mathbf{P}X} \arrow[dl, "\mathbf{P}X"] \\
& A &
\end{tikzcd}$$

But we could also define $\mathbf{P}X$ by using the construction we have used to obtain $\mathbf{Re}X$, replacing $\mathbf{Re}(\overline{X})$ and $\mathbf{Ce}(\overline{X})$ by $\mathbf{P}(\overline{X})$ and $\mathbf{C}(X)$. It is easy to show that both approaches yield the same result, and that in the second definition the quotiented version of Ψ is a real cleavage for a real Grothendieck bifibration.

We are left to show that the fibered versions of $\mathbf{Re}, \mathbf{P}$ are stable under change of base/context. This follows trivially from their fiberwise nature.

4.3. The Identity Type.

We now have all machinery necessary to model the Martin-Löf identity type rules. Let X be a small category. Since $\langle \partial_0, \partial_1 \rangle \colon \mathbf{P}(X) \to X \times X$ is a bifibration, obviously ∂_1 is one too. We already know a structure Ψ for it: it is "one half" of Equations (11–14). That is, given a path $\mathbf{p} \in \mathbf{P}(X)$, whose endpoint $\mathbf{p}_{\mathsf{e}} = \partial_1 \mathbf{p}$ we call y, along with $s \colon x \to y$ and $r \colon y \to z$, we have

$$\begin{aligned} r_* \mathbf{p} &= [r] * \mathbf{p} & \qquad s^* \mathbf{p} &= [\![s]\!] * \mathbf{p} \\ \Psi^r_*(y) &= \langle 1_y, r] * \mathbf{p} & \qquad \Psi^*_s(y) &= [\![s, 1_y\rangle * \mathbf{p} \,. \end{aligned} \tag{23}$$

(for simplicity we are are using the same notation for the denizens of $\mathbf{P}$ as we did for those of $\mathbf{Re}$).

Definition 8. *Let X be a small category and $Y \subseteq \mathbf{P}(X)$ a full subcategory of its category of paths. A* triangulator *is a map of categories $T : Y \to \mathbf{P}(\mathbf{P}(X))$ which satisfies the two equations*

$$\partial_1(T\mathbf{f}) = \mathbf{f}, \quad \partial_0(T\mathbf{f}) = \langle \partial_0 \mathbf{f} \rangle \tag{24}$$

for any map $\mathbf{f}$ in Y.

As an example of a triangulator we take

$$T\mathbf{f} = \mathbf{f} \star_\Psi \langle \partial_0 \mathbf{f} \rangle, \text{ which is a map in } \mathbf{P}(\mathbf{P}(X)). \tag{25}$$

Let us show what this looks like in practice. if $\mathbf{f}$ is the path

$$x \xrightarrow{a} \cdot \xrightarrow{b} \cdot \xleftarrow{c} \cdot \xrightarrow{d} \cdot \xleftarrow{e} y$$

then $T\mathbf{f}$ looks like

$$\begin{tikzcd}
x \arrow[d, equal] \arrow[dr, "a"] & & & & & \\
\cdot \arrow[r] \arrow[d, equal] & \cdot \arrow[d, equal] \arrow[dr, "b"] & & & & \\
\cdot \arrow[r] \arrow[d, equal] & \cdot \arrow[r] \arrow[d, equal] & \cdot \arrow[d, equal] & & & \\
\cdot \arrow[r] \arrow[d, equal] & \cdot \arrow[r] \arrow[d, equal] & \cdot \arrow[d, equal] & \cdot \arrow[l] \arrow[ul, "c"'] \arrow[d, equal] \arrow[dr, "d"] & & \\
\cdot \arrow[r] \arrow[d, equal] & \cdot \arrow[r] \arrow[d, equal] & \cdot \arrow[d, equal] & \cdot \arrow[l] \arrow[r] \arrow[d, equal] & \cdot \arrow[d, equal] & \\
x \arrow[r, "a"'] & \cdot \arrow[r, "b"'] & \cdot & \cdot \arrow[l, "c"'] \arrow[r, "d"'] & \cdot & y \arrow[l, "e"'] \arrow[ul, "e"']
\end{tikzcd}$$

where every horizontal combination of squares + one triangle is a map of paths.

Let now $X : \overline{X} \to A$ be a fibration, $(Z, \underline{Z}) : \overline{Z} \longrightarrow \overline{\mathbf{P}X}$ an arbitrary type, and $t : \overline{X} \to \overline{Z}$ be just as in Equation (4), i.e. $Z(t(s)) = \langle s \rangle$ for every map $s \in X$. Let also T be a triangulator defined on $\overline{\mathbf{P}X} \subseteq \mathbf{P}(\overline{X})$. Given a map $\mathbf{f}$ in $\mathbf{P}X$ we define

$$\big(J_X(Z,t)\big)\,\mathbf{f} \;=\; \partial_1\big(T\mathbf{f} \star_Z t(\partial_0 \mathbf{f})\big)\ . \tag{26}$$

This is well defined because $Z\langle t(\partial_0 \mathbf{f})\rangle = \langle \partial_0 \mathbf{f} \rangle = \partial_0(T\mathbf{f})$. We need to show this is a section of Z, but

$$\begin{array}{rcll}
Z\left(\partial_1\big(T\mathbf{f} \star_Z t(\partial_0\mathbf{f})\big)\right) & = & \partial_1\left(\mathbf{P}(Z)\big(T\mathbf{f} \star_Z t(\partial_0\mathbf{f})\big)\right) & \text{by naturality of } \partial_1 \\
& = & \partial_1(T\mathbf{f}) & \text{by def. of Hur. Action} \\
& = & \mathbf{f} & \text{by def. of triangulator.}
\end{array}$$

We also have to show this is an extension of t. Let $\mathbf{f} = \langle s \rangle$ for s in X. Notice that

$$T\langle s \rangle \;=\; \langle s \rangle \star_\Psi \langle \partial_0 \langle s \rangle \rangle \;=\; \langle s \rangle \star_\Psi \langle s \rangle \;=\; \langle\langle s \rangle\rangle\ ,$$

the latter equation because of (16). Then

$$\big(J_X(Z,t)\big)\langle s\rangle \;=\; \partial_1\big(T\langle s\rangle \star_Z t(\partial_0\langle s\rangle)\big) = \partial_1\big(\langle\langle s\rangle\rangle \star_Z t(s)\big) \;=\; \partial_1\langle t(s)\rangle \;=\; t(s)\;.$$

We are left to show that there are triangulators that are stable under change of base/context. But this is quite trivial in the case of the one given by Equation (25).

The action of $J_X(Z,t)$ is easy to describe on objects. First, start with paths of length one: let us denote the source and target of a map s in $\overline{X}$ by $\mathsf{d}_0 s, \mathsf{d}_1 s$ respectively. Then it is easy to see that

$$\big(J_X(Z,t)\big)\,[s] = \langle 1, s]_*\big(t(\mathsf{d}_0 s)\big) \quad\text{and}\quad \big(J_X(Z,t)\big)[\![s]\!] = [\![s,1\rangle^*\big(t(\mathsf{d}_1 s)\big)\,. \tag{27}$$

For longer paths is just use induction. The calculations for maps are a little more involved, naturally.

5. Some calculations

Iterating the operator $\mathbf{P}$ on a fibration $X\colon \overline{X} \to A$ naturally produces a globular object:

$$\cdots \quad \overline{\mathbf{PPP}X} \underset{\partial_1\mathbf{PP}X}{\overset{\partial_0\mathbf{PP}X}{\rightrightarrows}} \overline{\mathbf{PP}X} \underset{\partial_1\mathbf{P}X}{\overset{\partial_0\mathbf{P}X}{\rightrightarrows}} \overline{\mathbf{P}X} \underset{\partial_1 X}{\overset{\partial_0 X}{\rightrightarrows}} \overline{X}$$

which is equiped with an "augmentation" X. Since $\mathbf{P}$ is equipped with its own internal category structure, and since everything happens in Cat, this globular structure in our specific model is actually a higher-dimensional cubical category.

In general, given a model of the identity type, a certain measure of "how degenerate" it is is given by observing how large n has to be for the internal graph

$$\overline{\mathbf{P}^{\,n}X} \xrightarrow{\;\mathbf{P}^{\,n}X\;} \overline{\mathbf{P}^{\,n-1}X \times_{\overline{\mathbf{P}^{\,n-2}X}} \mathbf{P}^{\,n-1}X} \tag{28}$$

to be degenerate in some sense. Here $\mathbf{P}^{\,0}X = \overline{X}, \mathbf{P}^{\,-1}X = A$. For example in the first ever model which was not completely degenerate [7], the equivalent to $\mathbf{P}^{\,2}X$ is a discrete graph (i.e., isomorphic to the diagonal). Another form of degeneracy is when $\mathbf{P}^{\,n}X$ a relation, i.e., (28) is a monomorphism. A comprehensive taxonomy of these situations is given in [19]. This could be called an "absolute" test of degeneracy; under this test our model comes out with no degeneracy in any dimension.

Another way to measure degeneracy, which could be called "relative", is to consider the internal higher categorical structure which is deduced from the identity type axioms, and which in general is a kind of weak ω-groupoid. There have been several descriptions of the specific king of weak ω-groupoid structure which is obtained in general from identity types [4, 3, 13]. For example, although the inverse operation is always strictly involutive, composition in general is not strictly associative and strict associativity in a model is a form of degeneracy.

The only completely nondegenerate model has been constructed so far is the Kan complex model [15, 18, 16][3].

We want to give an impression of where our model stands in these hierarchies. Intuitively, since it is based on Grothendieck fibrations, which are defined trough a universal property (i.e., not *uniquely*, but up to *unique* isomorphism), relative degeneracy should begin at dimension 3, and this is what indeed happens. In what follows we will show the rudiments of the computations that are necessary to assert this.

In the presence of dependent products, the set of rules we have given for the identity type $\mathbf{P}X$ associated to a dependent type X can be generalized to a "parametrized" version as follows: Let $(W, \underline{W})\colon \overline{W} \to \overline{\mathbf{P}X}$ be a type, $(Z, \underline{Z})\colon \overline{Z} \to \overline{W}$ another one, and let now $t\colon \overline{\mathsf{r}^*W} \to \overline{Z}$ be such that $Z \circ t = h$, where h is the upper part of the pullback:

$$\begin{tikzcd}
 & \overline{Z} \arrow[d, "Z"'] \\
\overline{\mathsf{r}^*W} \arrow[ur, "t"] \arrow[r, "h"'] \arrow[d, "\mathsf{r}^*W"'] & \overline{W} \arrow[u, bend right, "J_X(W;Z,t)"'] \arrow[d, "W"] \\
\overline{X} \arrow[r, "\mathsf{r}"] \arrow[dr, "\Delta"'] & \overline{\mathbf{P}X} \arrow[d, "\mathbf{P}X"] \\
 & \overline{X \times_A X}
\end{tikzcd}$$

Then the parametrized version of the identity rules requires that there be a section $J_X(W; Z, t)$ that extends t. This generalization is easy to prove using Π; it is also the way to define an identity type in the absence of Π (the true general definition replaces the single type W by an arbitrary sequence $\overline{W}_n \to \overline{W}_{n-1} \to \cdots \to \overline{W}_1 \to \overline{\mathbf{P}X}$ but our seemingly more restricted version is enough for the purposes of this illustrative section, and is equivalent anyway because our model has the sum operator Σ, as the interested reader can show easily).

Let now w be an object of $\overline{W}$, which is above the path $\mathbf{p}$. The discussion right above allows us to assume that $\mathbf{p}$ has length one, since we can use induction to deduce the general case. So we put $\mathbf{p} = [s]$ for a map in $\overline{X}$. We know that objects of $\overline{\mathsf{r}^*W}$ are pairs (x, y), where x is an object of $\overline{X}$ and y an object of W which is above $\langle x \rangle$.

It is not hard to see, using the calculations displayed in Equations (7) and (27) that

$$J_X(W; Z, t)w \quad = \quad \underline{Z}_*^{W^*_{\langle 1,s]}(w)} \big(t(\mathsf{d}_0 s, \langle 1, s]^* w)\big)$$

[3] That was true at the time this paper was submitted. We are aware that since then newer models that are nondegenerate in that sense have appeared.

in more detail:

$$\begin{array}{ccc} t\left(\mathsf{d}_0 s, \langle 1,s]^* w\right) & \longrightarrow & \underline{Z}_*^{\underline{W}^*_{\langle 1,s]}(w)}\left(t(\mathsf{d}_0 s, \langle 1,s]^* w)\right) \\ \langle 1,s]^* w & \xrightarrow{\underline{W}^*_{\langle 1,s]}(w)} & w \\ \langle \mathsf{d}_0 s\rangle & \xrightarrow[\langle 1,s]]{} & [s] \end{array} \tag{29}$$

and thus symmetrically, putting $\mathbf{p} = [\![s]\!]$

$$J_X(W;Z,t)[\![s]\!] \quad = \quad \underline{Z}^*_{\underline{W}_*^{[\![s,1\rangle}(w)}\left(t(\mathsf{d}_1 s, [\![s,1\rangle_* w)\right) \ .$$

These calculations simplify considerably when Z, W are types that depend only on $\overline{X \times_A X}$ (i.e. obtained by pulling back with $\mathbf{P}X$). One important case is the substitution rule for equality. Let $(Y, \underline{Y})\colon \overline{Y} \to \overline{X}$ be a type, and denote by Y_0 the pullback of Y by the first projection $\overline{X \times_A X} \to \overline{X}$ and by Y_1 the pullback of Y by the second one. For W, Z now take $W = (\mathbf{P}X)^* Y_0$ and $Z = (\mathbf{P}X \circ W)^* Y_1$. An object of $\overline{W}$ can be presented as a pair $(\mathbf{q}, y)$ where $y \in \overline{Y}$ is above $\partial_0 \mathbf{q}$, while an object of $\overline{Z}$ as a triple $(\mathbf{q}, y, z)$, where $(\mathbf{q}, y)$ is just as above and $z \in Y$ is above $\partial_1 \mathbf{q}$. Given maps of paths $\mathbf{f}\colon \mathbf{p} \to \mathbf{q}$ and $\mathbf{g}\colon \mathbf{q} \to \mathbf{r}$ then according to (5) and (6) the cartesian and cocartesian maps associated to $(\mathbf{q}, y)$ are

$$(\mathbf{p}, (\partial_0 \mathbf{f})^* y) \xrightarrow{(\mathbf{f}, \underline{Y}^*_{\partial_0 \mathbf{f}}(y))} (\mathbf{q}, y) \quad \text{ and } \quad (\mathbf{q}, y) \xrightarrow{(\mathbf{g}, \underline{Y}^{\partial_1 \mathbf{g}}_*(y))} (\mathbf{r}, (\partial_0 \mathbf{g})_* y)$$

respectively. Much the same goes with the fibration Z: let now $(\mathbf{f}, u)\colon (\mathbf{p}, x) \to (\mathbf{q}, y)$ and $(\mathbf{g}, r)\colon (\mathbf{q}, y) \to (\mathbf{r}, w)$ be two maps in $\overline{W}$, and $(\mathbf{q}, y, z)$ an object in $\overline{Z}$. The cartesian $\underline{Z}^*_{(\mathbf{f},u)}(\mathbf{q}, y, z)$ and cocartesian $\underline{Z}^{(\mathbf{g},r)}_*(\mathbf{q}, y, z)$ are

$$(\mathbf{p}, x, (\partial_1 \mathbf{f})^* z) \xrightarrow{(\mathbf{f}, u, \underline{Y}^*_{\partial_1 \mathbf{f}}(z))} (\mathbf{q}, y, z) \quad \text{ and } \quad (\mathbf{q}, y, z) \xrightarrow{(\mathbf{g}, r, \underline{Y}^{\partial_1 \mathbf{g}}_*(z))} (\mathbf{r}, w, (\partial_1 \mathbf{g})^* y)$$

respectively.

Now, given the above data, and seeing an object of $\overline{\mathsf{r}^* W}$ as a pair (x, y) where $x \in \overline{X}$ and $y \in \overline{Y}$ is above $x = \partial_0 \langle x \rangle$, the substitution rule for equality on the type Y is obtained directly from $J_X(W;Z,t)$ when

$$t(x,y) = (\langle x\rangle, y, y) \ . \tag{30}$$

Choosing an arbitrary map s in $\overline{X}$ and object $([s], y)$ in $\overline{W}$, and taking into account the fact that $\partial_0 \langle 1, s] = 1_{\mathsf{d}_0 s}, \partial_1 \langle 1, s] = s$, we get

$$J_X(W;Z,t)([s], y) = ([s], y, s_*(1^* y))$$

where, in more details, Diagram (29) becomes

$$\begin{array}{ccc} ((\langle \mathsf{d}_0 s\rangle, 1^* y, 1^* y) & \xrightarrow{(\langle 1,s], \underline{Y}_1^*(y), \underline{Y}^s_*(1^* y))} & ([s], y, s_*(1^* y)) \\ (\langle \mathsf{d}_0 s\rangle, 1^*_{\mathsf{d}_0 s} y) & \xrightarrow{(\langle 1,s], \underline{Y}_1^*(y))} & ([s], y) \\ \langle \mathsf{d}_0 s\rangle & \xrightarrow[\langle 1,s]]{} & [s] \end{array}$$

and symmetrically

$$J_X(W;Z,t)(\llbracket s\rrbracket, y) = (\llbracket s\rrbracket, y, s^*(1_* y))\ .$$

It is well-known that in a model of identity types every $\mathbf{P}X$ has a canonical weak ω-groupoid structure. Standard calculations tell us that its internal 1-composition is obtained by substituting for $(Y,\underline{Y})$ in the above the type (∂_1,Ψ) that was described at the beginning of Section 4.3. In other words, given paths $\mathbf{p},\mathbf{q}$ such that $\partial_1\mathbf{p} = \partial_0\mathbf{q}$, denoting that internal composition operator by $\mathbf{q}\cdot\mathbf{p}$, those standard calculations say that

$$(\mathbf{p},\mathbf{q},\mathbf{q}\cdot\mathbf{p}) = J_X(W;Z,t)(\mathbf{p},\mathbf{q})$$

when $(Y,\underline{Y}) = (\partial_0 X,\Psi)$ and t is just as in (30). But, using Equation (23) we see that

$$[s]\cdot\mathbf{p} = [s]*[1]*\mathbf{p} \quad\text{and}\quad \llbracket s\rrbracket\cdot\mathbf{p} = \llbracket 1\rrbracket * \llbracket s\rrbracket * \mathbf{p}$$

Let $\mathbf{q} = \mathbf{q}_1 * \mathbf{q}_2 * \cdots \mathbf{q}_n$ be the uniquely defined decomposition of path $\mathbf{q}$ as a vertical composite of paths of length one. Applying induction on the length of $\mathbf{q}$, we see that $\mathbf{q}\cdot\mathbf{p} = \tilde{\mathbf{q}}_1 * \tilde{\mathbf{q}}_2 * \cdots * \tilde{\mathbf{q}}_n * \mathbf{p}$ where

$$\tilde{\mathbf{q}}_i \;=\; \begin{cases} [s_i]*[1] & \text{if}\quad \mathbf{q}_i = [s_i] \\ \llbracket 1\rrbracket * \llbracket s_i\rrbracket & \text{if}\quad \mathbf{q}_i = \llbracket s_i\rrbracket \end{cases}$$

Thus we immediately see that $\mathbf{r}\cdot(\mathbf{q}\cdot\mathbf{p}) = (\mathbf{r}\cdot\mathbf{q})\cdot\mathbf{p}$. But *this does not mean that 1-composition is strict.* This is because the real test of strictness is the path between $\mathbf{r}\cdot(\mathbf{q}\cdot\mathbf{p})$ and $(\mathbf{r}\cdot\mathbf{q})\cdot\mathbf{p}$ in $\mathbf{PP}X$, and further computations will show that this path is not the identity path, but a constant path of nonzero length. Thus the 2-cells of the weak ω-groupoid are not degenerate, but being constant paths, they are isomorphisms, and the cell structure in dimension 3 becomes truly degenerate.

Acknowledgements I would like to thank Christian Retoré and Thomas Streicher for reasons that they know.

[1] Awodey, S., Warren, M. A., 2009. Homotopy-theoretic models of identity types. Math. Proc. Camb. Phil. Soc. 146 (45), 45–55.

[2] Cartmell, J., 1986. Categorical models of dependent type theory. Annals of Pure and Applied Logic 32, 209–243.

[3] Gambino, N., Garner, R., 2008. The identity type weak factorisation system. Theoretical Computer Science 409, 94–109.

[4] Garner, R., Berg, B. v. d., 2011. Types are weak ω-groupoids. Proceedings of the London Mathematical Society 102 (2), 370–374.

[5] Garner, R., Berg, B. v. d., 2012. Topological and simplicial models of identity types. ACM Transaction on Computational Logic 13 (1), 3:1–3:44.

[6] Hofmann, M., 1995. On the interpretation of type theory in locally cartesian closed categories. In: Computer Science Logic (Kazimiers 1994). Vol. 933 of Lect. Notes in Comp. Sci. Springer, pp. 427–441.

[7] Hofmann, M., Streicher, T., 1998. The groupoid interpretation of type theory. In: Twenty-five Years of Constructive Type Theory (Venice 1995). Vol. 36 of Oxford Logic Guide Series. Oxford University Press, pp. 83–111.

[8] Jacobs, B., 1999. Categorical Logic and Type Theory. No. 141 in Studies in Logic. Elsevier.

[9] Lamarche, F., 1989. Modelling polymorphism with categories. Ph.D. thesis, Mc Gill University.

[10] Lamarche, F., 2013. Path functors in *Cat*, submitted, available at `http://hal.inria.fr/hal-00831430`.

[11] Lambek, J., 1994. Bilinear logic in algebra and linguistics. In: Girard, J., Lafont, Y., Regnier, L. (Eds.), Advances in Linear Logic. Vol. 222 of London Math. Soc. Lec. Notes. Cambridge University Press, pp. 43–59.

[12] Lawvere, W., 1973. Metric spaces, generalized logic and closed categories. Rendiconti del Seminario Matematico e Fisico di Milano XLIII, 135–166, available as a TAC reprint: `http://www.tac.mta.ca/tac/reprints/articles/1/tr1.pdf`.

[13] Lumsdaine, P., 2009. Weak ω-categories obtained from weak factorization systems. In: Typed Lambda-Calculi and Applications. Vol. 5608 of Lect. Notes in Comp. Sci.

[14] Pitts, A. M., 2000. Categorical logic. In: Abramsky, S., Gabbay, D., Maybaum, T. (Eds.), Handbook of Logic in Computer Science. Oxford University Press, pp. 39–128.

[15] Streicher, T., 2006. Identity types and weak omega-groupoids, notes of talk given at Uppsala, available at `http://www.mathematik.tu-darmstadt.de/~streicher/uppsala.pdf.gz`.

[16] Streicher, T., 2013. A model of type theory in simplicial sets: A brief introduction to Voevodskys homotopy type theory. J. of Applied Logic.

[17] Taylor, P., 1987. Recursive domains, indexed category theory and polymorphism. Ph.D. thesis, University of Cambridge.

[18] Voevodsky, V., 2006–2009. A very short note on homotopy lambda-calculus, available on `http://www.math.ias.edu/~vladimir/Site3/Univalent_Foundations_files/`.

[19] Warren, M. A., 2008. Homotopy theoretic aspects of constructive type theory. Ph.D. thesis, Carnegie-Mellon University.